OCR

RECOGNISING ACHIEVEMENT

General Certificate of Education
Advanced Supplementary (AS) and Advanced Level

former Oxford and Cambridge modular syllabus

MEI STRUCTURED MATHEMATICS 5504

Pure Mathematics 4

Thursday **11 JANUARY 2001** Morning 1 hour 20 minutes

Additional materials:
Answer paper
Graph paper
Students' Handbook

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

Write your name, Centre number and candidate number in the spaces provided on the answer paper/answer booklet.

Answer any **three** questions.

Write your answers on the separate answer paper provided.

If you use more than one sheet of paper, fasten the sheets together.

INFORMATION FOR CANDIDATES

The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.

You are advised that an answer may receive no marks unless sufficient detail of the working is shown on the answer paper to indicate that a correct method is being used.

This question paper consists of 3 printed pages and 1 blank page.

HN/2

[Turn over

1 A curve has equation $y = \dfrac{(x-6)(x+2)}{(2x-3)(x+1)}$.

(i) Write down the equations of the three asymptotes. [3]

(ii) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. Hence show that (0, 4) is a stationary point, and find the coordinates of the other stationary point. [6]

(iii) Sketch the curve. [4]

(iv) On a separate diagram, sketch the curve with equation $y^2 = \dfrac{(x-6)(x+2)}{(2x-3)(x+1)}$.

Give the coordinates of the stationary points on this curve, and the equations of the asymptotes. [7]

2 (a) Solve the inequalities

(i) $\dfrac{x+3}{2x-1} \geqslant 2$,

(ii) $\dfrac{2x-1}{x+3} \leqslant \dfrac{1}{2}$. [7]

(b) (i) Prove by induction that $\sum_{r=1}^{n} r3^r = \frac{3}{4} + \frac{1}{4}(2n-1)3^{n+1}$. [7]

(ii) Hence find the sum of the first n terms of the series

$$1 \times 3 + 3 \times 3^2 + 5 \times 3^3 + 7 \times 3^4 + \ldots$$

giving your answer in its simplest form. [6]

3 The cubic equation $z^3 + 6z^2 + 12z + 16 = 0$ has one real root α and two complex roots β, γ.

(i) Verify that $\alpha = -4$, and find β and γ in the form $a + bj$. (Take β to be the root with positive imaginary part.) [5]

(ii) Find $\frac{1}{\beta}$ and $\frac{1}{\gamma}$ in the form $a + bj$. [3]

(iii) Find the modulus and argument of each of α, β and γ. [4]

(iv) Illustrate the six complex numbers $\alpha, \beta, \gamma, \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ on an Argand diagram, making clear any geometrical relationships between the points. [5]

(v) Find $\arg(\beta - \alpha)$. Draw on your diagram the locus L of points representing complex numbers z for which $\arg(z - \alpha) = \arg(\beta - \alpha)$. [3]

4 **(i)** Find the vector product $\begin{pmatrix} 3 \\ -2 \\ -18 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$. [3]

(ii) Find the equation of the line of intersection of the two planes

$$3x - 2y - 18z = 6$$
$$2x + y - 5z = 25.$$ [4]

You are given the matrix equation

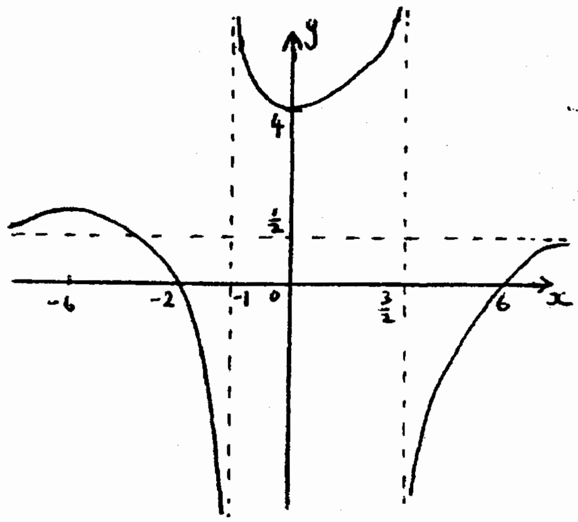
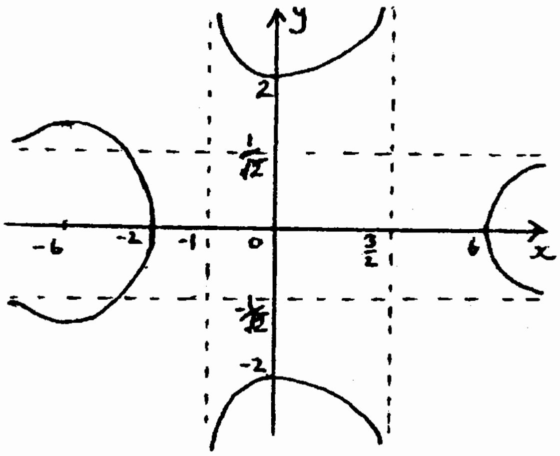
$$\begin{pmatrix} 3 & -2 & -18 \\ 2 & 1 & -5 \\ 7 & k & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 25 \\ 20 \end{pmatrix}. \qquad (*)$$

(iii) Solve the equation (*) when $k = -32$. [5]

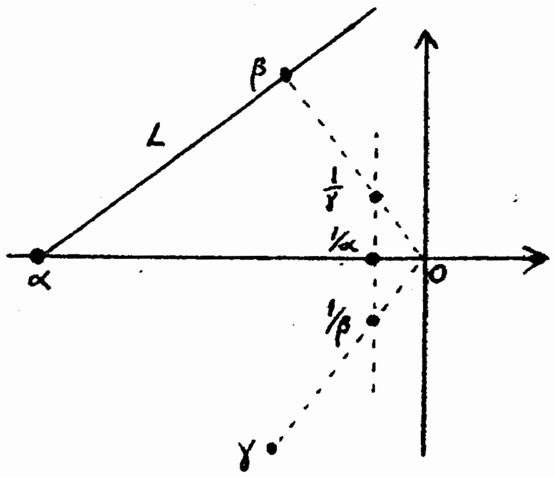
(iv) Find the value of k for which the equation (*) does not have a unique solution. Determine whether there is no solution or whether there are infinitely many solutions. Give a geometrical interpretation. [8]

Mark Scheme

MARKING SCHEME

1 (i)	$x=\frac{3}{2},\quad x=-1,\quad y=\frac{1}{2}$	B1B1B1 **3**	Maximum B2 may be implied from graph or by $y\to\frac{1}{2}$ etc
(ii)	$\frac{dy}{dx}=\frac{(2x^2-x-3)(2x-4)-(x^2-4x-12)(4x-1)}{(2x-3)^2(x+1)^2}$	M1 A1	Attempt to use quotient rule (or equivalent) Any correct form
	$=\frac{7x^2+42x}{(2x-3)^2(x+1)^2}$		
	$\frac{dy}{dx}=0$ when $7x^2+42x=0$	M1	Obtaining a quadratic equation
	Stationary points when $x=0,\ y=4$	B1 ft	$x=0$ obtained or verified
	and $x=-6,\ y=\frac{16}{25}=0.64$	M1 A1 cao **6**	Solving quadratic to obtain a value of x
(iii)		B1 B1 B1 B1 **4**	Curve passing through $(-2,0)$ with negative gradient, and through $(6,0)$ with positive gradient Central section (minimum on y-axis) Clear maximum on LH branch Curve of correct shape approaching all asymptotes correctly
(iv)		B3	Award B1 (ft) for every <u>two</u> of: • Infinite gradient at $(-2,0)$ • No curve in $-2<x<-1$ • No curve in $\frac{3}{2}<x<6$ • Infinite gradient at $(6,0)$ • Symmetry in the x-axis (one section) • Symmetry in the x-axis (other two) *Max 2 if not fully correct shape*
	Four stationary points $(0,\ \pm2)$, $\left(-6,\ \pm\frac{4}{5}\right)$	B2 ft	Give B1 for any two correct
	Four asymptotes $x=\frac{3}{2},\quad x=-1,\quad y=\pm\frac{1}{\sqrt{2}}$	B2 ft **7**	Give B1 for any two correct *Max B1B1 can be implied from numbers written on axes*

2 (a)(i)	$\dfrac{x+3}{2x-1} = 2$ when $x = \frac{5}{3}$ $\dfrac{x+3}{2x-1} \geq 2$ when $\frac{1}{2} < x \leq \frac{5}{3}$	B1 M1 A2 cao **4**	For $\frac{5}{3}$ or factor $(5-3x)$ *Can be earned in (ii) if not given in (i)* Considering intervals defined by critical values $\frac{1}{2}$, $\frac{5}{3}$ (ft) Give A1 for < instead of ≤ etc
(ii)	$\dfrac{2x-1}{x+3} \leq \dfrac{1}{2}$ when $-3 < x \leq \frac{5}{3}$	M1 A2 ft **3**	Considering intervals defined by critical values -3, $\frac{5}{3}$ (ft) Give A1 for < instead of ≤ etc, but penalise this only once
(b)(i)	When $n=1$, LHS = 3 RHS $= \frac{3}{4} + \frac{1}{4} \times 1 \times 3^2 = 3$ Assuming it is true for $n = k$, $\sum_{r=1}^{k+1} r3^r = \frac{3}{4} + \frac{1}{4}(2k-1)3^{k+1} + (k+1)3^{k+1}$ $= \frac{3}{4} + \frac{1}{4} \times 3^{k+1}(2k - 1 + 4k + 4)$ $= \frac{3}{4} + \frac{1}{4} \times 3^{k+1} \times 3(2k+1)$ $= \frac{3}{4} + \frac{1}{4}(2k+1)3^{k+2}$ True for $n = k \Rightarrow$ True for $n = k+1$ Hence true for all positive integers *n*, by induction	B1 M1A2 M1 A1 A1 **7**	For using $3^{k+1} \times 3 = 3^{k+2}$ Correctly obtained For completion *Dependent on all previous marks*
(ii)	Series is $\sum (2r-1)3^r$ $= \sum 2r3^r - \sum 3^r$ $\sum_{r=1}^{n} 2r3^r = \frac{3}{2} + \frac{1}{2}(2n-1)3^{n+1}$ $\sum_{r=1}^{n} 3^r = \dfrac{3(3^n - 1)}{3-1}$ $\sum_{r=1}^{n} (2r-1)3^r = \frac{3}{2} + \frac{1}{2}(2n-1)3^{n+1} - \frac{3}{2}(3^n - 1)$ $= 3 + n3^{n+1} - 3^{n+1}$	B1 M1 A1 M1A1 A1 cao **6**	
	OR $\sum (2r-1)3^r$ B1 $= \sum r3^r + \sum (r-1)3^r$ M1 $= \sum_{r=1}^{n} r3^r + 3\sum_{r=1}^{n} (r-1)3^{r-1}$ $= \sum_{r=1}^{n} r3^r + 3\sum_{r=0}^{n-1} r3^r$ M1 $= \frac{3}{4} + \frac{1}{4}(2n-1)3^{n+1} + 3\left\{\frac{3}{4} + \frac{1}{4}(2n-3)3^n\right\}$ A1A1 $= 3 + 3^{n+1}(n-1)$ A1		*Dependent on M1M1*

3 (i)	When $z=-4$, LHS $=-64+96-48+16=0$ So -4 is a root	B1	Or showing that $(z+4)$ is a factor
	Equation is $(z+4)(z^2+2z+4)=0$	M1 A1	Factorising the cubic For z^2+2z+4
	Complex roots given by $z^2+2z+4=0$ $z=\dfrac{-2\pm\sqrt{-12}}{2}=\dfrac{-2\pm j\sqrt{12}}{2}$	M1	Solving quadratic and $\sqrt{-n}=j\sqrt{n}$
	$\beta=-1+j\sqrt{3}$, $\gamma=-1-j\sqrt{3}$	A1 cao 5	Accept $-1\pm1.73j$
	OR Roots are -4, $\beta=a+bj$, $\gamma=a-bj$ where $-4+2a=-6$, $-4(a^2+b^2)=-16$ M2 $a=-1$, $b=\sqrt{3}$ A1A1		
(ii)	$\dfrac{1}{\beta}=\dfrac{1}{-1+j\sqrt{3}}\times\dfrac{-1-j\sqrt{3}}{-1-j\sqrt{3}}$	M1	
	$=-\dfrac{1}{4}-\dfrac{\sqrt{3}}{4}j$	A1 ft	
	$\dfrac{1}{\gamma}=-\dfrac{1}{4}+\dfrac{\sqrt{3}}{4}j$	A1 ft 3	ft conjugate of $\dfrac{1}{\beta}$
(iii)	$\lvert\alpha\rvert=4$, $\arg\alpha=\pi$ (*or* 180° *or* 3.14)	B1	
	$\lvert\beta\rvert=\lvert\gamma\rvert=2$	B1 ft	
	$\arg\beta=\frac{2}{3}\pi$ (*or* 120° *or* 2.1)	B1 cao	
	$\arg\gamma=-\frac{2}{3}\pi$ (*or* $-120°$ *or* -2.1 *or* 240° *or* $\frac{4}{3}\pi$ *or* 4.2)	B1 ft 4	ft $-\arg\beta$
(iv)		B3 ft	Six points in approx correct positions (Give B1 for two correct B2 for four correct)
		B1	For $\dfrac{1}{\gamma}$, $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$ in a vertical line
		B1 5	*Either* O, $\dfrac{1}{\gamma}$, β *or* O, $\dfrac{1}{\beta}$, γ in a straight line *Max 4 if not on a single diagram*
(v)	$\arg(\beta-\alpha)=\frac{1}{6}\pi$ (*or* 30° *or* 0.52)	B1 ft	
		B1 ft	Any part of line through α and β or a half line starting at α in approx correct direction
		B1 ft 3	Line starting at α passing through and extending beyond β

4 (i)	$\begin{pmatrix}3\\-2\\-18\end{pmatrix}\times\begin{pmatrix}2\\1\\-5\end{pmatrix}=\begin{pmatrix}28\\-21\\7\end{pmatrix}$	B1 B1 B1 3	Max 2 if final answer given as $\begin{pmatrix}4\\-3\\1\end{pmatrix}$
(ii)	When $z=0$, $3x-2y=6$ and $2x+y=25$ $x=8,\ y=9$ One point on the line is $(8,\ 9,\ 0)$ Line of intersection is $\mathbf{r}=\begin{pmatrix}8\\9\\0\end{pmatrix}+\lambda\begin{pmatrix}4\\-3\\1\end{pmatrix}$	M1 A1 M1 A1 ft 4	Finding one point on line of intersection Or $(20, 0, 3)$ or $(0, 15, -2)$ etc Using direction from (i) *or* finding a second point *Accept any form*
	OR $7x-28z=56$, so $x=8+4z$ $y=25-2x+5z=9-3z$ M2 $x=8+4\lambda,\ y=9-3\lambda,\ z=\lambda$ A1A1		Obtaining (e.g.) x and y in terms of z Give A1 for two correct
(iii)	$7(8+4\lambda)-32(9-3\lambda)+2\lambda=20$ $\lambda=2$	M1A1 ft M1	 Solving to find λ
	OR M1 $7y+21z=63,\quad -82y+132z=18$ A1 M1		Eliminating one variable in two ways Solving to find at least two variables
	OR $\mathbf{M}^{-1}=\frac{1}{882}\begin{pmatrix}-158 & 580 & 28\\-39 & 132 & -21\\-71 & 82 & 7\end{pmatrix}$ M1 M1 A1		Attempt at adjoint matrix Dividing by determinant Correct inverse
	$x=16,\ y=3,\ z=2$	A2 cao 5	Give A1 (ft) if just one slip made
(iv)	$7(8+4\lambda)+k(9-3\lambda)+2\lambda=20$ No unique solution when $28-3k+2=0$	M1 M1	
	OR $y+3z=9,\ (3k+14)y+132z=18$ M1 Not unique when $3k+14=44$ M1		Eliminating one variable in two ways
	OR Det$=3(2+5k)+2(4+35)-18(2k-7)$ M1 Not unique when $210-21k=0$ M1		
	OR Not unique if $\begin{pmatrix}7\\k\\2\end{pmatrix}\times\begin{pmatrix}3\\-2\\-18\end{pmatrix}=\mu\begin{pmatrix}4\\-3\\1\end{pmatrix}$ M1 $6+126=-3\mu,\quad -14-3k=\mu$ M1		Considering $\begin{pmatrix}7\\k\\2\end{pmatrix}\times\begin{pmatrix}3\\-2\\-18\end{pmatrix}$
	$k=10$ Consistent if $56+9k=20$; not true when $k=10$ Hence there are no solutions Planes form a (triangular) prism *Or* Line of intersection of two planes is parallel to the third plane *etc*	A1 cao M2 A1 B2 8	 Checking consistency by any valid method; *must use RHS* Deduced from fully correct working *Dependent on 'No solution'*

Examiner’s Report

1. 9.46×10^{20}m.

Pure Mathematics 4 (5504)

General Comments

There were many excellent scripts, with about 20% of candidates scoring 50 marks or more (out of 60), and a wide spread of marks, with about a quarter scoring less than 30. Some may have found the paper a little long, but this was usually caused by the use of overcomplicated methods. On the other hand, many candidates attempted at least part of a fourth question. Questions 1 and 2 were somewhat more popular than questions 3 and 4.

Comments on Individual Questions

Question 1 (Curve sketching)

This was the best answered question, with half the attempts scoring 16 marks or more (out of 20). In part (i) almost all candidates gave the three asymptotes correctly. In part (ii), most candidates applied the quotient rule correctly to obtain the derivative, but then very many made errors (usually of signs) when multiplying out and simplifying, so that the second stationary point was often wrong. In part (iii) the curve was usually sketched well; the main reason for loss of marks was failure to show the maximum on the left-hand branch with the curve approaching the horizontal asymptote from above, even when this stationary point had been found correctly in part (ii). The square root graph in part (iv) was well understood, although marks were often lost for not showing clearly that the curve is vertical where it crosses the x-axis, for omitting one or more of the negative sections, and for not listing all four stationary points and all four asymptotes.

(i) $x=\frac{3}{2}$, $x=-1$, $y=\frac{1}{2}$; (ii) $\dfrac{7x^2+42x}{(2x-3)^2(x+1)^2}$, $(-6, 0.64)$;

(iv) $(0, \pm 2)$, $\left(-6, \pm\frac{4}{5}\right)$; $x=\frac{3}{2}$, $x=-1$, $y=\pm\dfrac{1}{\sqrt{2}}$.

Question 2 (Inequalities and series)

A very substantial number did not have a valid method for solving the inequalities in part (a); the effect of a zero or negative denominator was frequently ignored, and some began by squaring both sides. By contrast, the proof by induction in part (b)(i) was very well understood. In the final part (b)(ii), most candidates could write the series as $\sum(2r-1)3^r = 2\sum r3^r - \sum 3^r$; but then $\sum 3^r$ was rarely recognised as a geometric series, so only a few completed this successfully. A fairly common approach was to assume that $\sum 3^r = \dfrac{\sum r3^r}{\sum r}$.

(a)(i) $\frac{1}{2} < x \leq \frac{5}{3}$; (ii) $-3 < x \leq \frac{5}{3}$; (b)(ii) $3 + 3^{n+1}(n-1)$.

Question 3 (Complex numbers)

This question was quite well answered, with half the attempts scoring 14 marks or more. In part (i), most candidates were able to demonstrate that -4 was a root and obtained the quadratic equation $z^2 + 2z + 4 = 0$. However, many did not solve this correctly, making sign errors or taking $\sqrt{-12}$ to be $12\mathrm{j}$ instead of $\sqrt{12}\,\mathrm{j}$. In part (ii) the method for dividing complex numbers was well understood and usually carried out correctly. In part (iii), most candidates found the modulus and argument of the complex roots correctly, although the real root -4 caused some trouble. In part (iv), plotting the points on an Argand diagram was done well, except for α and $\dfrac{1}{\alpha}$ which were often plotted on the positive real axis and sometimes on the imaginary axis. The geometrical relationships (inverses on a vertical line, and 0, $\dfrac{1}{\gamma}$, β in a straight line) were often not clearly shown. In part (v), the correct value of $\arg(\beta - \alpha)$ was surprisingly rare, and only a few drew the locus correctly; circles, and straight lines through the origin, were fairly common here.

(i) $\beta = -1 + \mathrm{j}\sqrt{3}$, $\gamma = -1 - \mathrm{j}\sqrt{3}$; (ii) $\dfrac{1}{\beta} = -\dfrac{1}{4} - \dfrac{\sqrt{3}}{4}\mathrm{j}$, $\dfrac{1}{\gamma} = -\dfrac{1}{4} + \dfrac{\sqrt{3}}{4}\mathrm{j}$;
(iii) $|\alpha| = 4$, $|\beta| = |\gamma| = 2$, $\arg\alpha = \pi$, $\arg\beta = \frac{2}{3}\pi$, $\arg\gamma = -\frac{2}{3}\pi$; (v) $\frac{1}{6}\pi$.

Question 4 (Vectors and matrices)

This was the worst answered question, with half the attempts scoring 10 marks or less. Most candidates found the vector product correctly in part (i), and used it to find the line of intersection in part (ii). However, only a minority went on to use this line of intersection to solve the equations in part (iii); the most common approach was elimination, and some used the inverse matrix. Small slips in arithmetic and algebra were very common here, often resulting in very complicated fractions. Part (iv) turned out to be quite demanding, and many candidates did not attempt it. There were some very efficient solutions using the line of intersection from part (ii). However, it was more common for the determinant to be used; although this often gave the correct value of k, further progress was rare, with careless slips sometimes leading to a unique solution being found.

(i) $\begin{pmatrix} 28 \\ -21 \\ 7 \end{pmatrix}$; (ii) $\mathbf{r} = \begin{pmatrix} 8 \\ 9 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$; (iii) $x = 16$, $y = 3$, $z = 2$; (iv) $k = 10$, no solution.